MATH 1650: CHAPTER 4: ROOT / RADICAL AND POWER FUNCTIONS WORKSHEET

MATH 1650 REVIEW OF ROOTS AND RADICALS

DEFINITION: For a real number a, and a natural number $n \geq 2$, an n^{th} **root of** a is a number x where $x^n = a$.

EXAMPLE:

- $x = 4$ and $x = -4$ are both second (square) roots of 16 since $(4)^2 = 16$ and $(-4)^2 = 16$.
- $x = 2$ is a third (cube) root of 8 since $(2)^3 = 8$.
- The n^{th} roots of 0 are 0.

NOTE: In general, given any real number a:

- if n is **odd**, there is exactly one real n^{th} root of a.
- if n is **even**:
  - if $a < 0$, there are no real n^{th} roots of a.
  - if $a = 0$, the n^{th} root of a is 0.
  - if $a > 0$, there are exactly two real n^{th} roots of a: one root is positive; one root is negative.

DEFINITION: Given a real number a, the **principal** n^{th} **root of** a, denoted $\sqrt[n]{a}$ is defined as follows:

- if n is odd: $\sqrt[n]{a}$ is the n^{th} root of a (there is only one if n is odd, so this is the one!)
- if n is even: $a \geq 0$ and we choose $\sqrt[n]{a}$ so that $\sqrt[n]{a} \geq 0$. (we choose the positive one.)
  NOTE: for $n = 2$, we write $\sqrt{a}$ to mean $\sqrt[2]{a}$.

EXAMPLE:

- Even though $x = 4$ and $x = -4$ are both square roots of 16, $\sqrt{16} = 4$ (the positive root.)
- Since 2 is the only real number with $(2)^3 = 8$, $\sqrt[3]{8} = 2$
- Even though both $x = 3$ and $x = -3$ are both fourth roots of 81, $\sqrt[4]{81} = 3$ (the positive root.)
- Since -2 is the only real number with $(-2)^5 = -32$, $\sqrt[5]{-32} = -2$.
- Since $-16 < 0$, $\sqrt[4]{-16}$ is not a real number. **NOTE:** $\sqrt{-16} = 4i$, but $\sqrt[4]{-16} \neq 2i$. Why?
- $\sqrt[n]{0} = 0$ for all n.

PROPERTIES OF RADICALS: If $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are real numbers and m is a natural number:

- **PRODUCT RULE:** $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
- **QUOTIENT RULE:** $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$, provided $y \neq 0$.
- **POWER RULE:** $\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$
- **INVERSE PROPERTY:**
  - For all natural numbers n (even and odd), $\left(\sqrt[n]{x}\right)^n = x$
  - If n is **odd**, $\sqrt[n]{x^n} = x$
  - if n is **even**, $\sqrt[n]{x^n} = |x|$

NOTE: In general, $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$

DEFINITION OF RATIONAL EXPONENTS: Let a be a real number, m an integer and n a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ and is defined whenever $\sqrt[n]{a}$ is defined.
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$, whenever $\left(\sqrt[n]{a}\right)^m$ is defined.

EXAMPLE: True or false? $\left(x^2\right)^{\frac{1}{2}} = x$

HINT: False! Why? $\left(x^2\right)^{\frac{1}{2}} = \sqrt{x^2} = |x|$

NOTE: In general, when raising an even power to an even root, you will need to simplify using absolute values.

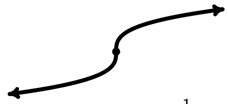
HELPFUL ADVICE: *WHEN IT DOUBT, WRITE IT OUT!* **EXAMPLE:** Solve: $x^{2/3} = 4$:

- Method One: $x^{2/3} = 4 \implies \sqrt[3]{x^2} = 4 \implies x^2 = 4^3 \implies x^2 = 64 \implies x = \pm 8$.
- Method Two: $x^{2/3} = 4 \implies (x^{2/3})^{3/2} = 4^{3/2} \implies |x| = 8 \implies x = \pm 8$.

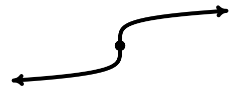
ZERO PROPERTY: If $p > 0$, then $u^p = 0 \implies u = 0$.

ROOT, RADICAL, AND RATIONAL EXPONENT FUNCTIONS

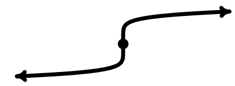
ODD INDEX ROOT FUNCTIONS: When n is odd, the graph of $f(x) = \sqrt[n]{x}$ resembles:



$y = \sqrt[3]{x} = x^{\frac{1}{3}}$



$y = \sqrt[5]{x} = x^{\frac{1}{5}}$

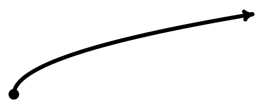


$y = \sqrt[7]{x} = x^{\frac{1}{7}}$

As n increases, the curve gets 'steeper' at the origin and 'flatter' as $x \to \pm\infty$.

When n is odd, the domain of $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

EVEN INDEX ROOT FUNCTIONS: When n is even, the graph of $f(x) = \sqrt[n]{x}$ resembles:



$y = \sqrt{x} = x^{\frac{1}{2}}$

$y = \sqrt[4]{x} = x^{\frac{1}{4}}$

$y = \sqrt[6]{x} = x^{\frac{1}{6}}$

As n increases, the curve gets 'steeper' at the origin and 'flatter' as $x \to \infty$.

When n is even, the domain of $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ is $[0, \infty)$ and the range is also $[0, \infty)$.

EXAMPLE: Find the domains of the following functions. Write your answers using interval notation.

- $f(x) = \sqrt{4-x}$

  Owing to the square root, we need $4 - x \geq 0$ so $x \leq 4$. Hence the domain is $(-\infty, 4]$.

- $g(x) = (4-x)^{2/3}$

  Rewriting $g(x) = \left(\sqrt[3]{4-x}\right)^2$ we see we have an odd-indexed root, so there are no domain restrictions.

  Hence the domain is $(-\infty, \infty)$.

- $F(t) = \dfrac{1}{2 - \sqrt[5]{t+3}}$

  The root here is an odd-indexed root, so we don't get any restrictions from the root itself.

  However, we do have a denominator, so to find the excluded values we solve $2 - \sqrt[5]{t+3} = 0$.

  We get $\sqrt[5]{t+3} = 2$ so $\left(\sqrt[5]{t+3}\right)^5 = 2^5$. Hence, $t + 3 = 32$ giving us the excluded value: $t = 29$.

  Our domain is $\{t \mid t \neq 29\}$ or, in interval notation: $(-\infty, 29) \cup (29, \infty)$.

- $G(t) = \dfrac{1}{2 - (t+3)^{3/2}}$

  We first note that $(t+3)^{3/2} = \left(\sqrt{t+3}\right)^2$, so because of the square root, $t + 3 \geq 0$ so $t \geq -3$.

  We also have a denominator to worry about, To find the excluded values, we solve $2 - (t+3)^{3/2} = 0$

  We get $(t+3)^{3/2} = 2$. Rewriting, we get $\left(\sqrt{t+3}\right)^3 = 2$ so $\sqrt{t+3} = \sqrt[3]{2}$.

  Squaring both sides, we get $\left(\sqrt{t+3}\right)^2 = \left(\sqrt[3]{2}\right)^2$ so $t + 3 = \sqrt[3]{4}$ or $t = -3 + \sqrt[3]{4}$.

  Hence, our domain is: $\left\{t \mid t \geq -3 \,\text{and}\, t \neq -3 + \sqrt[3]{4}\right\}$ or $\left[-3, -3 + \sqrt[3]{4}\right) \cup \left(-3 + \sqrt[3]{4}, \infty\right)$.

EXAMPLE: Find the domains of the following functions.

- $f(x) = \sqrt{2x - 5}$

- $g(x) = \dfrac{2x - 1}{\sqrt[3]{x - 4}}$

- $F(t) = (3t - 1)^{\frac{3}{4}}$

- $G(t) = \dfrac{2t}{4 - \sqrt{t + 9}}$

EXAMPLE: Find the domains of the following functions. Write your answers using interval notation.

- $h(x) = \sqrt{x-5} + \sqrt{4-x}$

  Owing to the $\sqrt{x-5}$, we need $x - 5 \geq 0$ so $x \geq 5$. Because of the $\sqrt{4-x}$, we need $4 - x \geq 0$ so $x \leq 4$.

  Since are no real numbers x which satisfy both $x \geq 5$ and $x \leq 4$, there is no domain for h.

  We write that the domain of h is the empty set, denoted $\varnothing$.

- $T(x) = (x-5)^{2/5} + (4-x)^{2/5}$

  Rewriting $(x-5)^{2/5} = \left(\sqrt[5]{x-5}\right)^2$ and $(4-x)^{2/5} = \left(\sqrt[5]{4-x}\right)^2$, we see we have odd-indexed roots.

  Hence we have no domain restrictions, so the domain is $(-\infty, \infty)$.

- $H(t) = \sqrt{4-t^2}$

  Because of the square root, we need $4 - t^2 \geq 0$. We have two ways to solve this inequality:

  METHOD 1: $4 - t^2 \geq 0$ so $4 \geq t^2$ or $t^2 \leq 4$. Hence, $\sqrt{t^2} \leq \sqrt{4}$ so $|t| \leq 2$.

  Since $|t| \leq 2$ is equivalent to $-2 \leq t \leq 2$, the domain is $[-2, 2]$.

  METHOD 2: We solve $4 - t^2 \geq 0$ using a Sign Diagram. Let $f(t) = 4 - t^2$ so we are looking for $f(t) \geq 0$.

  We find $f(t) = 0$ when $4 - t^2 = 0$ or $t^2 = 4$. Hence, $t = \pm 2$. Making a Sign Digram, we find:

  $$\begin{array}{cccccccc} & (-) & 0 & (+) & 0 & (-) & f(t) \\ -\infty & \longleftarrow & + & \longrightarrow & + & \longrightarrow & \infty \\ & & -2 & & 2 & & t \end{array}$$

  We find $f(t) \geq 0$ on $[-2, 2]$, so our domain is $[-2, 2]$.

- $v(x) = \left(\dfrac{2x-1}{x+5}\right)^{3/4}$

  Rewriting $v(x) = \left(\dfrac{2x-1}{x+5}\right)^{3/4} = \left(\sqrt[4]{\dfrac{2x-1}{x+5}}\right)^3$ we see that because of the 4th root, we need $\dfrac{2x-1}{x+5} \geq 0$.

  We let $f(x) = \dfrac{2x-1}{x+5}$ and make a Sign Diagram for $f(x)$.

  We find f is undefined when $x + 5 = 0$ or $x = -5$. Solving $f(x) = 0$ gives $2x - 1 = 0$ so $x = \frac{1}{2}$.

  Making our Sign Diagram, we find:

  $$\begin{array}{cccccccc} & (+) & ? & (-) & 0 & (+) & f(x) \\ -\infty & \longleftarrow & + & \longrightarrow & + & \longrightarrow & \infty \\ & & -5 & & \frac{1}{2} & & x \end{array}$$

  Our domain is where $f(x) \geq 0$ which is $(-\infty, -5) \cup \left[\frac{1}{2}, \infty\right)$.

EXAMPLE: Find the domain of the following functions.

- $f(x) = \sqrt{x^2 - x - 6}$

- $g(x) = \left(\dfrac{4-x}{x+1}\right)^{3/2}$

- $h(t) = 2t\,(t^2 - 1)^{-2/3}$

  HINT: $2t\,(t^2-1)^{-2/3} = \dfrac{2t}{(t^2-1)^{2/3}}$.

EXAMPLE: Make a sign diagram for the following functions.

- $f(x) = \dfrac{2x-3}{\sqrt{x+5}}$

  The first step in making a Sign Diagram for these sorts of function is to find the domain.

  Because of $\sqrt{x+5}$, we require $x+5 \geq 0$ or $x \geq -5$.

  Since $\sqrt{x+5}$ is in the denominator, we know $x \neq -5$ or otherwise we'd have 0 there.

  Hence, our domain of f is $\{x \,|\, x > -5\}$ or $(-5, \infty)$.

  Next, we find the zeros of f. Solving $f(x) = 0$ gives $2x - 3 = 0$ so $x = \frac{3}{2}$. Our Sign Diagram is:

  $$\begin{array}{ccccc} ? & (-) & 0 & (+) & \\ \vdash & \text{---} & + & \text{---} & \rightarrow \\ -5 & & \frac{3}{2} & & \end{array}$$

- $g(x) = x^{2/3}(1-x)^{-5/3}$

  Rewriting, we find $g(x) = \dfrac{x^{2/3}}{(1-x)^{5/3}}$.

  We note that since $x^{2/3} = \left(\sqrt[3]{x}\right)^2$ and $(1-x)^{5/3} = \left(\sqrt[3]{1-x}\right)^5$, we have no restrictions due to the radicals.

  We do have a denominator, so we solve for the excluded values: $(1-x)^{5/3} = 0$ so $1 - x = 0$ or $x = 1$.

  Our domain is thus $\{x \,|\, x \neq 1\}$ or $(-\infty, -1) \cup (-1, \infty)$.

  To find the zeros of g, we solve $g(x) = 0$ which amounts to solving $x^{2/3} = 0$. We get $x = 0$.

  Our Sign Diagram is hence:

  $$\begin{array}{cccccccc} & (+) & 0 & (+) & ? & (-) & g(x) \\ -\infty \leftarrow & \text{---} & + & \text{---} & + & \text{---} & \rightarrow \infty \\ & & 0 & & 1 & & x \end{array}$$

EXAMPLE: Make a sign diagram for $f(x) = (x+2)(1-x)^{-3/2}$.

EXAMPLE: Solve the following inequality using a Sign Diagram: $9(x-2)^{-1/3} \leq 3x(x-2)^{-4/3}$

As usual, our first step is to rewrite the inequality in the form $f(x) \leq 0$ and construct a Sign Diagram.

$$\begin{aligned}
9(x-2)^{-1/3} &\leq 3x(x-2)^{-4/3} \\
9(x-2)^{-1/3} - 3x(x-2)^{-4/3} &\leq 0 \\
\frac{9}{(x-2)^{1/3}} - \frac{3x}{(x-2)^{4/3}} &\leq 0 && \text{Rewrite negative exponents.} \\
\frac{9}{(x-2)^{1/3}} \cdot \frac{(x-2)^{3/3}}{(x-2)^{3/3}} - \frac{3x}{(x-2)^{4/3}} &\leq 0 && \text{Get a common denominator.} \\
\frac{9(x-2)^{3/3}}{(x-2)^{1/3+3/3}} - \frac{3x}{(x-2)^{4/3}} &\leq 0 \\
\frac{9(x-2)}{(x-2)^{4/3}} - \frac{3x}{(x-2)^{4/3}} &\leq 0 \\
\frac{9x-18}{(x-2)^{4/3}} - \frac{3x}{(x-2)^{4/3}} &\leq 0 \\
\frac{9x-18-3x}{(x-2)^{4/3}} &\leq 0 \\
\frac{6x-18}{(x-2)^{4/3}} &\leq 0
\end{aligned}$$

We let $f(x) = \dfrac{6x-18}{(x-2)^{4/3}}$ and look to solve $f(x) \leq 0$ using a Sign Diagram.

Since $(x-2)^{4/3} = \left(\sqrt[3]{x-2}\right)^4$, we have no domain restrictions due to the radical.

Solving the denominator, $(x-2)^{4/3} = 0$, we get $x - 2 = 0$ or $x = 2$ as our excluded value.

To find the zeros of f, we solve $f(x) = 0$. We get $6x - 18 = 0$ or $x = 3$.

Our Sign Diagram is:

$$\begin{array}{ccccccc}
 & (-) & ? & (-) & 0 & (+) & g(x) \\
-\infty & \longleftarrow & 2 & \text{———} & 3 & \longrightarrow & \infty \\
 & & & & & & x
\end{array}$$

We see $f(x) \leq 0$ on $(-\infty, 2) \cup (2, 3]$, so this is our final answer.

EXAMPLE: Solve $x\,(x-3)^{-3/2} \geq 4(x-3)^{-1/2}$

HOMEWORK: Section 4.1: 13 - 25 odd; Section 4.2: 9 - 15 odd, 18 - 20 all; Section 4.3: 1 - 31 odd.